# Cyclotomic splitting fields for projective Schur algebras over number fields

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Abstract Let $k$ be a number field and for every positive integer $m$ denote by $\xi_m$ a root of unity of order $m$ in an algebraically closed field extension $C$ of $k$. Let $A$ be a simple $k$-algebra wich occurs as a simple component of a twisted group algebra $(k, G, f)$ where $G$ is a finite group and $f : G \times G \to k^*$is a central 2-cocycle. Denote by $K \subset C$ the center of $A$. The purpose of this note is to construct a cyclotomic splitting field for $A$ of the form $K(\xi_m)$ where $m$ is determined from properties of $K$,$f$ and of a certain simple component of the ordinary group algebra of some finite central group extension of the commutator subgroup of $G$. The proof is based on results in representation theory, in cohomology and in the theory of algebras which are all well known.

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**§1. Clifford reduction to the commutator subgroup**

Let $k$ be a field of characteristic 0 and let $C/k$ be a field extension which is algebraically closed. Let $G$ be a finite group and let $f : G \times G \to k^*$ be a central 2-cocycle. Then the twisted group algebra $(k, G, f)$ is the associative $k$-algebra which is generated as a $k$-vector space freely by symbols $b_x, x \in G$, and whose ring multiplication is defined by extending the relations $b_x b_y = f(x, y)b_{xy}$, $x, y \in G$; $ab_x = b_x a$, $x \in G$, $a \in k$; $k$-linearly. As is well known, this $k$-algebra is semisimple. For this and other basic results on twisted group algebras, projective representations and related facts about cocycles see $[Y]$. Let $\chi$ be a $C-$irreducible $f-$character of $G$ and denote by $A = A(G, \chi)$ the corresponding simple component of $(k, G, f)$; its center is $K = k(\chi)$, the field which is generated over $k$ by all values of $\chi$ ; this follows from the more general result $[B]$, Satz 3. As is well known, see e.g. $[O]$, (4.2), the restriction $f' = f_{|G'}$ of $f$ to the commutator subgroup $G'$ is cohomologous over $k$ to a central 2-cocycle $f' : G' \times G' \to W_k$ , where $W_k$ denotes the group of roots of unity of $k$. Denote by $H$ a finite central group extension of $G'$ which lifts $f'$, i.e. such that the inflation of $f'$ to $H$ is cohomologically trivial. Then there is a $k$-algebra epimorphism of the ordinary group algebra $(k, H)$ onto $(k, G', f')$. Let $\varphi$ be a $C$-irreducible $f'$-character of $G'$

which is a constituent of the restriction of $\chi$ to $G'$, and denote by $A' = A(G', \varphi)$ the simple component of $(k, G', f')$ - and therefore of $(k, H)$ - corresponding to $\varphi$. The center of $A'$ is $K' = K(\varphi)$. $K'$ is a Galois ebtension of $K$ because it is contained in a cyclotomic extension of $K$. $K'$ is uniquely determined by $A$; this follows from Clifford theory for cocycle characters which can be developed along the lines in $[C]$; see also $[M], [R]$. For every central simple $k$-algebra $B$ denote by $e_k(B)$ its exponent, i.e. the order of the similarity class of $B$ in the Brauer group of $k$.

**(1.1) Proposition** *The field $K'$ contains a root of unity of order $e_{K'}(K' \otimes_K A)$.*

In the proof we will make use of *symbol algebras* $A_\omega(a, b)$ where $a, b \in K$ and $\omega$ is a primitive root of unity of order $m$ in $K$; these atre central simple $K-$algebras generated by symbols $X, Y$ which satify the relations

$$X^m = a \cdot 1,\ Y^m = b \cdot 1,\ YX = \omega XY \quad ,$$

and which are split over $K$ if and only if $a$ is a norm in the Kummer extension $K(\sqrt[m]{b})/K$ ; it follows that the exponent of $A_\omega(a, b)$ divides $m$; for details see $[ML]$, § 15.

Proof of (1.1): Denote by $I \leq G$ the inertia subgroup of $\varphi$, i.e.
$I = \{g \in G : \varphi^g = \varphi\}$, where
$\varphi^g(x) := \alpha_f(g, x)\varphi(gxg^{-1})$, $\alpha_f(g.x) := f(g, x)f(gx, g^{-1})/f(g, g^{-1})$
for all $g \in G$, $x \in G'$; this definition is suggested by the formular
$b_g b_x b_g^{-1} = \alpha_f(g, x) b_{gxg^{-1}}$ , $g \in G$, $x \in G'$.
$I$ is a normal subgroup of $G$ because the factor group $G/G'$ is abelian. As in $[O]$, section 4, especially (4.1), we obtain a central 2-cocycle $t : I/G' \times I/G' \to K'^*$, the so called *Clifford cocycle*, a $C$-irreducible $t$-character $\rho$ of $I/G'$ such that the simple component $A(I/G', \rho)$ of $(K', I/G', t)$ has center $K'$ and a similarity of $K'$-algebras

**(1.2)** $K' \otimes_K A \sim (K' \otimes_K A(G', \varphi)) \otimes_{K'} A(I/G', \rho)$.

Since $A(G', \varphi)$ is a simple component of the ordinary group algebra $(k, H)$ the field $K'$ contains a root of unity of order $e_{K'}(A(G', \varphi))$, see $[J]$. As is well known $A(I/G', \rho)$ is similar over $K'$ to a tensor product of $K'$-central symbol algebras $A_1, ..., A_r$, and therefore, as is well known, see e.g. $[ML]$, §15, especially 15.1 in connection with 15.7, $K'$ contains a root of unity of order $e_{K'}(A_i)$ for every $i = 1, ..., r$ ; for details see e.g. $[O]$, section 3. The assertion follows.

**Remark** The projective Schur group $P(k)$ of $k$ is the subgroup of the Brauer group of $k$ which consists of all similarity classes of central simple $k$-algebras which can be represented by a simple component of a twisted group algebra

$(k, G, f)$ where $G$ is a finite group and $f : G \times G \to k^*$ is a central 2-cocycle; see $[LO]$. We note that for every positive integer $r$ the set $P_r(k)$ which consists of all elements in $P(k)$ which can be represented by a simple component of a twisted group algebra of the form $(k, G, f)$, where the exponent of the commutator subgroup of $G$ divides $r$, is a subgroup of $P(k)$, and we get a sequence of projective Schur groups $P_1(k), P_2(k), ...$ such that $P(k) = \cup_{r \in \mathbb{N}} P_r(k)$.

## §2. The case of number fields

Now let $k$ be a number field. For every finite subextension $M/k \subset C/k$ denote by $w_M$ the order of the group of roots of unity in $M$. From (1.1) and (1.2) we deduce

**(2.1) Proposition** *The exponent of* $A(G, \chi)$ *divides the number*
$t(k, \chi, G') := (K' : K) \cdot w_{K'}$
*where* $K = k(\chi)$, $K' = K(\varphi)$, $\varphi$ *denoting any irreducible constituent of the restriction of* $\chi$ *to* $G'$.

Lifting $\varphi$ to a character $\widetilde{\varphi}$ of a linear representation of a central extension of $G'$ with center $W_k$, and applying Brauer 's induction theorem to $\widetilde{\varphi}$ shows that the field $K(\xi_{w_k \cdot \exp(G')})$ contains $K'$ as a subfield and is a splitting field of $A(G', \varphi)$; see e.g. $[S]$, 12.3, Theorem 24. It follows that the exponent of $A(G, \chi)$ divides the number $(K(\xi_{w_k \cdot \exp(G')}) : K) \cdot w_{K(\xi_{w_k \cdot \exp(G')})}$.

So we conclude

**(2.2) Proposition** *If* $k$ *is a number field the exponent of the simple component* $A(G, \chi)$ *of* $(k, G, f)$ *divides a number which depends only on* $k$, $K = k(\chi)$ *and the exponent of the commutator subgroup* $G'$ *of* $G$.

For any finite extension $L/k \subset C/k$ and any central 2-cocycle $g : G \times G \to L^*$ denote by $S(L, g)$ the finite set of places which divide the values of $g$ with nonzero multiplicity and which divide the order of $G$ and the infinite places of $L$. Let $S(K, \chi)$ denote the finite set of places of $K$ such that $A$ splits outside $S(K, \chi)$.

**(2.3) Proposition** $S(K, \chi) \subset S(K, f)$

Proof:: The proof is along the lines of the proof in the case $f = 1$ in $[D]$, VII, §5, Satz 3. The discriminant of any maximal order of $A(K, \chi)$ over the ring $R_S$ of $S$-integers of $K$, where $S = S(K, f)$, divides the discriminant of $(K, G, f)$ because $A(K, \chi)$ is generated over $K$ by the $T(x)$, $x \in G$, where $T$ is any $C$-irreducible $f$-representation with character $\chi$ and because
$T(x)^{order(x)} = \prod_{i=1}^{order(x)} f(x, x^i) \cdot T(e) \in R_S^* \cdot Id$
for all $x \in G$, which shows that the $T(x)$, $x \in G$, are integral over $R_S \cdot Id$. The discriminant of $(K, G, f)$ is - up to sign - given by

$|G|^{|G|}\prod_{x\in G} f(x,x^{-1})$ ;
see e.g. $[Y]$, section 4.2. In view of $[D]$, VI, §6, Satz 2, this implies that $A(K,\chi)$ splits outside $S(K,f)$.

Now we use a standard argument, see e.g. $[D]$, VII, §5, proof of Satz 4, or $[T]$, p.192, Lemma, to obtain a cyclotomic splitting field for $A(K,\chi)$: We choose $m$ large enough such that the local degrees at all places in $S(K,f)$ of the extension $K(\xi_m)/K$, where $\xi_m \in C$ is a root of unity of order $m$, are divisible by the number $t(k,\chi,G')$ which was defined in (2.1). Then by the theory of central simple algebras over local number fields, especially $[D]$, VII, §2, Satz 4, and by the local global principle in the theory of central simple algebras over number fields, especially $[D]$, VII, §5, we obtain the following proposition.

**(2.4) Proposition** $K(\xi_m)$ is a cyclotomic splitting field of $A(k,\chi)$.

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